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ON THE INITIAL SINGULARITY PROBLEM IN TWO-DIMENSIONAL QUANTUM COSMOLOGY

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Abstract

The problem of how to put interactions in two-dimensional quantum gravity in the strong coupling regime is studied. It shows that the most general interaction consistent with this symmetry is a Liouville term that contain two parameters (α, β) satisfying the algebraic relation $2\beta - \alpha = 2$ in order to assure the closure of the diffeomorphism algebra.

The model is classically soluble and it contains as general solution the temporal singularity. The theory is quantized and we show that the propagation amplitude fall to zero in $\tau = 0$. This result shows that the classical singularities are smoothed by quantum effects and the bing-bang concept could be considered as a classical extrapolation instead of a physical concept.

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The quantization of the gravitational field is a problem that has resisted a solution for many years is spite of the intense research in this field [1].

The black holes evaporation [2], loss of information [3] or the quantum non-hair theorems [4] are ideas that emerged in the last two decades for which there are not still a definitive explanation in terms of a true quantum theory of gravity.

However in spite of these results are very promisories, fundamentals problems associated to the physical interpretation of the theory are very far to be reached and it seems to indicate that is necessary to introduce significatives simplifications in order to get a complete understanding of the theory.

In [5] we proposed a two-dimensional quantum gravity model in the strong coupling regime (SCR) which was mapped to a null string theory embeded in a two-dimensional target space. In this model, the temporal component of the bi-vector on the target Φ_A was identified as the time in quantum gravity and, as a consequence, one could define an evolution operator or a probability concept as in ordinary quantum mechanics.

The purpose of the present letter is to extend our previous analysis in order to include interactions in the SCR respecting all the symmetries of the model. In particular, we will show below that the most general interaction consistent with the reparametrization invariance is a Liouville-like one, this interaction is very interesting because will permit to make contact with classical and quantum cosmology.

More precisely, we will show that the general solutions of the gravitational field equations are singular in $\tau = 0$, *i.e.* in two dimensions there is a Kasner-like solution although it has not an oscillatory behaviour as in four dimensions. The theory also can be exactly quantized and the propagation amplitude reflects the fact at quantum level, the classical singularities disappears[1].

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[1]Here one could ask if this result remain in other 2D gravity models. The answer is yes although the demostration must be performed model by model (an exception is the Jackiw model)

In the strong coupling limit $2D$ gravity is described by the constraints

$$\mathcal{H}_\perp = \frac{1}{2}\Big[b^2 - 4P^2\Big],$$
$$\mathcal{H}_1 = b\psi^{'} + P\mathcal{H}i^{'} - 2P^{'} + b^{'}, \tag{1}$$

where (b, ψ) and $(P, \mathcal{H}i)$ are canonical variables and the constraints (1) satisfy the diffeomorphism algebra [2]

$$[\mathcal{H}_\perp(x), \mathcal{H}_\perp(x^{'})] = 0,$$
$$[\mathcal{H}_\perp(x), \mathcal{H}_1(x^{'})] = (\mathcal{H}_\perp(x) + \mathcal{H}_\perp(x^{'}))\delta^{'}(x - x^{'}), \tag{2}$$
$$[\mathcal{H}_1(x), \mathcal{H}_1(x^{'})] = (\mathcal{H}_1(x) + \mathcal{H}_1(x^{'}))\delta^{'}(x - x^{'}).$$

The next step is try to put interactions. One can use as a guide principle that the possibles interactions in the model should respect the diffeomorphism symmetry, *i.e.* should preserve the algebra (2). This fact, say us that the most general constraints including interactions have the form

$$\mathcal{H}_\perp = \frac{1}{2}[b^2 - 4P^2] + \frac{1}{8}\omega e^{\alpha\psi + \beta\chi},$$
$$\mathcal{H}_1 = b\psi^{'} + P\chi^{'} - 2P^{'} + b^{'}, \tag{3}$$

where α, β and ω are constants. By a simple dimensional analysis one can see that ω is a constant that can be identified with the cosmological constant μ^2, whereas α and β are two constants that must be chosen satisfying the algebraic relation

$$2\beta - \alpha = 2, \tag{4}$$

in order to preserve the closure of the diffeomorphism algebra.

This result is remarkable because it says that one can fix appropiately α (or β) and to decouple the variables of the theory.

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[2]For conventions and notation see [5], however we will emphasize that the fields ψ and $\mathcal{H}i$ (in the proper time gauge are related with $g_{\mu\nu}$ by means $g_{00} = 1, g_{01} = g_{10} = 0, g_{11} = e^{\chi}$.

Such is discussed in [5] the action

$$S \;=\; \int d^2x\; [b\dot{\psi} + P\dot{\chi} - N\mathcal{H}_{\perp} - N_1\mathcal{H}_1], \tag{5}$$

is invariant under the following transformations

$$\begin{aligned}
&\delta\psi = \epsilon b + \epsilon_1\psi^{'} - \epsilon_1^{'},\\
&\delta\chi = -4\epsilon P + \epsilon_1\chi^{'} - 2\epsilon_1^{'},\\
&\delta b = -\alpha\epsilon\mu^2 e^{\alpha\psi+\beta\chi} + (\epsilon_1 b)^{'},\\
&\delta P = -\beta\epsilon\mu^2 e^{\alpha\psi+\beta\chi} + (\epsilon_1 P)^{'},\\
&\delta N = \dot{\epsilon} + N^{'}\epsilon_1 - N\epsilon_1^{'} + N^{'}\epsilon - N_1\epsilon^{'},\\
&\delta N_1 = \dot{\epsilon}_1 + N_1^{'}\epsilon_1 - N_1\epsilon_1^{'},
\end{aligned} \tag{6}$$

provided that $\epsilon(\tau_2, x) = 0 = \epsilon(\tau_1, x)$.

Using (5), the equations of motion after to eliminate the canonical momenta are

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\left[\frac{\dot{\psi}-N_1\psi^{'}+N_1^{'}}{N}\right] - \frac{\partial}{\partial x}\left[\frac{N_1(\dot{\psi}-N_1\psi^{'}+N_1^{'})}{N}\right] + \tfrac{1}{8}\mu^2 N e^{\alpha\psi+\beta\chi} = 0,\\
&\frac{\partial}{\partial\tau}\left[\frac{\dot{\chi}-N_1\chi^{'}-2N_1^{'}}{N}\right] - \frac{\partial}{\partial x}\left[\frac{N_1(\dot{\psi}-N_1\chi^{'}-2N^{'})}{N}\right] - \tfrac{1}{2}\mu^2 N e^{\alpha\psi+\beta\chi} = 0.
\end{aligned} \tag{7}$$

These equations are difficult to solve in an arbitrary gauge, but is suggested by a simple observation, one can solve it easily in the proper-time gauge [6]

$$\dot{N} = 0, \qquad N_1 = 0. \tag{8}$$

Using this gauge condition, (7) becomes

$$\begin{aligned}
&\ddot{\psi} + \tfrac{1}{8}\alpha\mu^2 N^2 e^{\alpha\psi+\beta\chi} = 0,\\
&\ddot{\chi} - \tfrac{1}{2}\mu^2\beta N^2 e^{\alpha\psi+\beta\chi} = 0.
\end{aligned} \tag{9}$$

It is interesting to note that similar equations appear also in gravity in four dimensions when the behaviour of the solutions of the Einstein field equations near of the temporal singularity is studied. In fact, the Einstein field equations for a homogeneous and anisotropic space are [7]

$$\ddot{a} = -\tfrac{1}{2}e^{4a}, \tag{10}$$

$$\ddot{b} = \ddot{c} = \tfrac{1}{2}e^{4a}, \tag{11}$$

where a, b and c are three functions that characterize the spatial metric tensor. The equation (10) coincides with the "Einstein equations" (9), while (11) has not analog in two dimensions. The set of equations (10-11) is responsible for the oscillatory regime near of the time singularity [8]. In our case at hand, the oscillatory regime is not possible due that there is not equation (11), however as we will see below even so a Kasner-like solution will be found.

The equations (9) can be written in the following way

$$\ddot{\rho} - \frac{1}{2}\gamma^2 e^{\rho} = 0, \tag{12}$$

where $\rho = \alpha\psi + \beta\chi$ and $\gamma^2 = \frac{1}{4}\mu^2 N^2(4\beta^2 - \alpha^2)$. The general solution is

$$\tau \;=\; \frac{1}{\gamma}\int \frac{d\rho}{\sqrt{D^2 + e^{\rho}}}, \tag{13}$$

where $D^2 = \frac{A}{\gamma^2}$ and A is an integration constant.

In order to integrate (13) three cases must be distinguished;

i) $D^2 < 0$, *i.e.* when $A > 0,\ \gamma^2 < 0$ or $A < 0,\ \gamma^2 > 0$ and (13) gives

$$\gamma\tau = c + \frac{2}{D}tan^{-1}\Big[\frac{\sqrt{D^2 + e^{\rho}}}{D}\Big], \tag{14}$$

ii) $D^2 > 0$, *i.e.* when $A > 0,\ \gamma^2 > 0$ or $A < 0,\ \gamma^2 < 0$ and (13) becomes

$$\gamma\tau = c + \frac{1}{D}\ln\Big[\frac{\sqrt{D^2 + e^{\rho}} - D}{\sqrt{D^2 + e^{\rho}} + D}\Big] \tag{15}$$

iii) $D^2 = 0$, *i.e.* $A = 0$ and (13) is

$$\tau = c - \gamma e^{-\rho/2}. \tag{16}$$

where c is a constant.

The cases considered above gives a relation between ψ and χ but the general solution of the equation of motion are obtained inserting this relation back in the equation of motion for ψ and χ.

The general solutions found are

$$\chi = -\frac{\mu^2 N^2 \beta}{2\gamma^2} \ln\cos\left(\frac{\gamma D}{2}\tau + c\right) - \frac{\mu^2 N^2 \beta D^2}{4}\tau^2 + \omega\tau + \sigma, \tag{17}$$

$$\chi = -\frac{2\mu^2 N^2 \beta}{\gamma^2} \ln\sinh\left(\frac{\gamma D}{2}\tau + c\right) + \omega\tau + \sigma, \tag{18}$$

$$\chi = -\frac{\mu^2 \beta N^2 \gamma^2}{2} \ln\left(c - \tau\right) + \omega\tau + \sigma, \tag{19}$$

for the models $i) - iii)$ and similar solutions for ψ (ω, σ are constants).

From these results one see that there are elections for α (or β) that gives singular solutions for $\tau = 0$, in fact choosing

$$\frac{\beta}{2\beta + \alpha} > 0, \qquad \beta(2\beta + \alpha) > 0, \tag{20}$$

and $\omega = 0 = \sigma$ for the models $ii)$ and $iii)$, one see that the general solution near of $\tau = 0$ in a synchronous frame has the form [3]

$$ds^2 = d\tau^2 - \frac{1}{\tau^p} dx_1^2, \tag{21}$$

whereas the general solution for the case $i)$ is regular everywhere independently of the values of α or β. The last point, has an analogue with some general regular cosmological solutions found in the literature [9].

Now we quantize the models discussed above; we assume as in [5] like-particle boundary conditions and, as a consequence, there are not anomalies in the functional measure.

The propagation amplitude in the proper-time gauge in the Euclidean space is

$$G[\chi_2, \chi_1; \psi_2, \psi_1] \; = \int_0^\infty \prod_x dN(x) \int \mathcal{D}\psi \mathcal{D}\chi e^{-\int d^2x [\frac{1}{2N}(\dot{\psi}^2 + \frac{1}{4}\dot{\chi}^2) + \frac{1}{8}\mu^2 N e^{\alpha\psi + \beta\chi}]}. \tag{22}$$

[3]It is remarkable to note that this solution corresponds to a Poincaré metric when $p = 2$ after to make the transformation $t = \sqrt{2\tau}$, i.e. a solution describing a space with negative constant curvature and with the condition $p = 2$ imposing a restriction on the possibles values of the cosmological constant. In three dimensions there is a similar solution [10] and it corresponds to an extreme black hole solution. I would like to thank J. Zanelli by describing these results to me.

This formula permits to determine the Green function for two metric configurations and one can interpret (22) as describing a set of infinite massless relativistic particles interacting with an external potential moving on a two dimensional target space.

In order to compute (22) one can use the "decoupling gauge" $\alpha = 0$, $\beta = 1$ and one obtain

$$G[\chi_2, \chi_1; \psi_2, \psi_1] \ = \prod_k^{\infty} \Big(\int_0^{\infty} dN_k N_k^{-\frac{1}{2}} e^{-\frac{(\Delta\psi_k)^2}{2N_k}} \mathcal{G}[\chi_{2k}, \chi_{1k}; N_k] \Big), \tag{23}$$

where $\mathcal{G}[\chi_{2k}, \chi_{1k}; N_k]$ is the propagation amplitude for Liouville quantum mechanics, namely

$$\mathcal{G}[\chi_{2k}, \chi_{1k}; N_k] = \int \mathcal{D}\chi_k e^{-\int d\tau (\frac{1}{8N_k}\dot{\chi}_k^2 + \frac{1}{8}\mu^2 N_k e^{\chi_k})}. \tag{24}$$

This formula is formally equivalent to an ordinary quantum mechanical problem described by the action

$$S_k = \int d\tau \Big(\frac{1}{8N_k}\dot{\chi}_k^2 + \frac{1}{8}\mu^2 N_k e^{\chi_k} \Big), \tag{25}$$

with $N_k^{-1}(0)$ playing the role of mass.

The calculation of (24) is performed more easily solving the Schrödinger equation

$$\Big[-\frac{d^2}{d\chi_k^2} - \mu^2 e^{\chi_k} \Big] \Psi[\chi_k] = p_k^2 \Psi[\chi_k] \tag{26}$$

where p_k is the momentum of the $k-th$ particle and inserting this solution in

$$\mathcal{G}\ [\chi_{2k}, \chi_{1k}; N_k] = \int_0^{\infty} dp_k e^{-2N_k p_k^2} \Psi^*(\chi_{2k}) \Psi(\chi_{1k}). \tag{27}$$

One can solve (26) distinguishing two cases *v.i.z.* $i)$ $\mu^2 < 0$ and $ii)$ $\mu^2 > 0$.

When $i)$ is satisfied the solution of the Schrödinger equation is [11], [12], [13]

$$\Psi[\chi_k] = \sqrt{\sinh 2\pi p_k} K_{2ip}(2\sqrt{\mu^2} e^{\chi_k/2}), \tag{28}$$

where K_{2ip} is the modified Bessel function.

Using (27) and (28), (23) become

$$G[\chi_2, \chi_1; \psi_2, \psi_1] = \prod_k^{\infty} \Big(\int_0^{\infty} dN_k N_k^{-1/2} e^{-(\Delta\psi_k)^2/2N_k} \times \\ \times \int_0^{\infty} dp_k e^{-2{p_k}^2 N_k} K^*_{2ip}(2\sqrt{\mu^2} e^{\chi_{2k}/2}) K_{2ip}(2\sqrt{\mu^2} e^{\chi_{1k}/2}) \Big), \tag{29}$$

The next question is, how one can know if (29) is a correct formula?. The answer is easily obtained observing that in the limit $\mu^2 \to 0$, it must reproduce the propagator for an infinite set of massless relativistic particles. In such limit, $K_\nu \sim x^{-\nu}$ and the expected result is obtained.

In (29), the exponential $e^{-(\Delta\psi)^2/2N} \to 0$ when $\tau \to 0$ due the solutions of the equations of motion are singular. The vanishing of this Green function say us that the temporal singularity never is reached and, in consequence, quantum mechanics smooths the classical singularities and the big-bang concept could, simply, be considered as a classical extrapolation.

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